THE PROCESSES AND PRODUCTS OF STUDENTS' GENERALIZING ACTIVITY

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Generalization has been a major focus of curriculum standards and research efforts in mathematics education. While researchers have documented many productive contexts for generalizing and the generalizations students make, less attention has been given to the processes of generalizing. Moreover, there has been less work done with high school students in advanced mathematical contexts. To address these issues we use a model of learning that enables us to make explicit the processes of generalizing. We exemplify this model of learning in the context of an interview study with high school students working on cubic relationships.

Keywords: Cognition, Learning Theory

Generalization has been a major focus of the Common Core State Standards as both a process and content standard. One reason for this focus in curriculum standards is that mathematics educators have identified that generalizing activity is a central means through which students construct new knowledge and as such should be a primary focus of school instruction (Davydov, 1972/1990). To date a majority of the research on generalization has taken place with elementary and early middle school students (grades K-7) in part because it is seen as a basis for and route to algebraic reasoning (e.g., Carraher, Martinez, & Schliemann, 2008). Furthermore, many of these studies have focused on patterns or functions that involve linear relationships, however, researchers have argued that there is a need for investigation of studying generalizing in situations that can involve non-linear relationships as well as studies that include older students (Amit & Neria, 2008). Dorfler (2008) has also identified that researchers studying generalization have focused their attention on developing contexts for generalization and characterizing the kinds and qualities of generalizations students make within these contexts. However, he identifies that there has been significantly less attention paid to the processes involved in generalizing (see Ellis, 2007 for an exception). To address these issues, we report on an interview study conducted with eight 10^{th}-12^{th} grade students who worked on establishing cubic relationships. We situate our work within a framework for studying learning where a central reason for this is so that we can specify particular processes involved in generalizing.

Theoretical Framework

Ellis (2007) differentiates between generalizing actions and reflection generalizations; a generalizing action is an action that precedes and may support a formal statement of generalization (i.e., is a process involved in generalizing) and a reflection generalization is a formal statement of generalization that a student expresses verbally or symbolically (i.e., a product of generalizing actions). We find this distinction useful in outlining our framework for learning which is based in scheme theory. A scheme is a repeatable way of operating that consists of three parts: an assimilatory mechanism, an activity, and a result (Piaget, 1970; Von Glasersfeld, 1995). The assimilatory mechanism involves a student in making an interpretation of a problem situation. The activity of a scheme involves the use of mental operations on imagined or perceptually present material, which transform the assimilated situation into a result. When a person re-processes the result of a scheme so that she can anticipate it prior to carrying out the activity, we call this an act of interiorization, and we consider the person to have constructed a concept. Concepts, then, are the results of schemes that are available to a person prior to operating and are what a person uses to assimilate situations.

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For a scheme to be repeatable in different experiential situations means that its formation entailed generalizing actions (Ellis, 2007): A person uses the same scheme in two or more different experiential situations, which implies that the person has abstracted some similarity among the experiential situations because they trigger, vis-à-vis assimilation, the same scheme. We make the distinction between assimilation and *generalizing* assimilation to mark when a generalizing action happens in a particular experiential situation (Piaget, 1970); assimilation entails the re-activation of a scheme, but the abstraction of similarity among current and past situations is not a primary feature of this re-activation. Assimilation is *generalizing* when a primary feature of the re-activation of a scheme is the abstraction of similarity among the current and prior situations. We consider this abstraction of similarity to be a generalizing action that entails an act of learning because a person modifies the assimilatory mechanism of a scheme.

One marker that distinguishes situations involving assimilation and those involving generalizing assimilation is that in the latter case a person may experience a perturbation, a sense of cognitive dissonance. This perturbation may be expressed as uncertainty about what to do to solve a situation even though from the perspective of an observer a student has a scheme that could be used to solve the situation. The resolution of this kind of perturbation can occur through an abstraction of similarity between the current and prior situations in which a student has used his or her scheme. We provide one empirical example of this kind of learning later in the paper.

We consider a second type of learning to be within the realm of generalizing actions as well. We call this type of learning a functional accommodation; a functional accommodation differs from a generalizing assimilation in that a student makes a modification to the *activity* of her scheme in the context of its use (Steffe, 1991). We specifically consider a functional accommodation to entail a generalizing action when it enables a student to solve a broader range of situations. We provide three empirical examples of this kind of learning later in the paper.

The claim that a student has engaged in a generalizing assimilation or a functional accommodation means a researcher is inferring that some modification that was novel occurred in the context of a student using his or her scheme. We highlight that such processes are in the province of reflective abstraction where a reflective abstraction involves a projection of a novel way of operating from a lower to higher level along with the reorganization of the novel way of operating at the higher level (Piaget, 1970). Here we consider the "lower" and "higher level" to be defined by the fact that the scheme itself becomes more general in nature either because the assimilatory structure is broadened to include new experiential situations or because the change in activity of a scheme allows a student to solve a broader range of problems.

None of the acts of learning described to this point necessarily involve a student in being consciously aware of having made a modification to his or her scheme. This is a key reason that we consider them to be in the realm of what Ellis (2007) calls generalizing actions—actions that precede a formal statement of generalization. Further, we note that generalizing actions can occur in the absence of an actual formal statement of generalization yet they are an important part of documenting the *processes* involved in generalizing. We consider formal statements of generalization, what Ellis calls reflection generalizations, to be in the province of a reflected abstraction (Piaget, 1970)—a retroactive thematization of a way of operating that brings this way of operating to conscious awareness. We consider reflection generalization to be in the province of a reflected abstraction because to make a formal statement of generalization entails becoming to some extent consciously aware of how one is operating. Moreover, as Ellis's definition suggests, a reflection generalizations involve symbolizing—formal statements of generalization are made either with natural language or mathematical notation (both symbol systems).

Following Von Glassersfeld (1995), we view symbols as involving bi-directional relationships among a sound/graphic image, a person's re-presentations, and a concept (p. 131). The most

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important point of Von Glassersfeld's model for this paper is that when a person has constructed a symbol, any one of the three (a sound/graphic image, a person's re-presentations, and a concept) can call up any of the others. This observation means that a sound/graphic image (e.g., a verbal statement or the letter "*x*") can be used to call forth a concept where a concept is the operations of a scheme that no longer need to be implemented either mentally or materially in order for a person to consider them to be part of an experiential situation. We provide one empirical example of a reflection generalization that involves the processes of symbolizing.

Methods and Methodology

We provide empirical examples of generalizing actions and reflection generalizations from an interview study conducted with eight 10^{th}-12^{th} grade students. We conducted two hour long video recorded interviews that focused on problems like the *Card Problem.*

> *Card Problem.* You have the 2, 3, and King of Diamonds, a friend has the 2, 3, and King of Hearts, and another friend has the 2, 3, and King of Clubs. A three-card hand consists of one card from each person's hand (order does not matter). How many different three card hands are possible to make? How many three-card hands have no face cards, exactly one face card, exactly two face cards, and exactly three face cards?

The aim of this problem was for students to develop the equivalence that $3^3 = (2 + 1)^3 = 2^3 + 3 \bullet (2^2 \bullet 1) + 3 \bullet (2 \bullet 1^2) + 1^3$. We conjectured that this equivalence could grow out of reasoning that there were a total of 3^3 possible three-card hands, that this total could be quantified as $(2 + 1)^3$ because each person had 2 non-face card and 1 face card, and also that this total could be quantified as $2^3 + 3 \bullet (2^2 \bullet 1) + 3 \bullet (2 \bullet 1^2) + 1^3$ because: the number of three card hands with no face cards was 2^3; there were 3 ways to have one face card with each way having $(2^2 \bullet 1)$ three card hands, etc. During the interviews, students were encouraged to represent this reasoning using a 3-D array that represented all possible three card hands (Figure 1a), and to identify different regions of this array that represented three card hands that had no face cards (Figure lb, green region), one face card (Figure 1b, three blue regions, only two visible), two face cards (Figure 1b, three yellow regions), and three face cards (Figure 1b, red region). Students then worked toward a statement of generalization that $(x + 1)^3 = x^3 + 3 \bullet (x^2 \bullet 1) + 3 \bullet (x \bullet 1^2) + 1^3$. We regard the examples as "learning in process" because it was not possible to determine from two interviews the extent to which the modifications that students made were lasting.

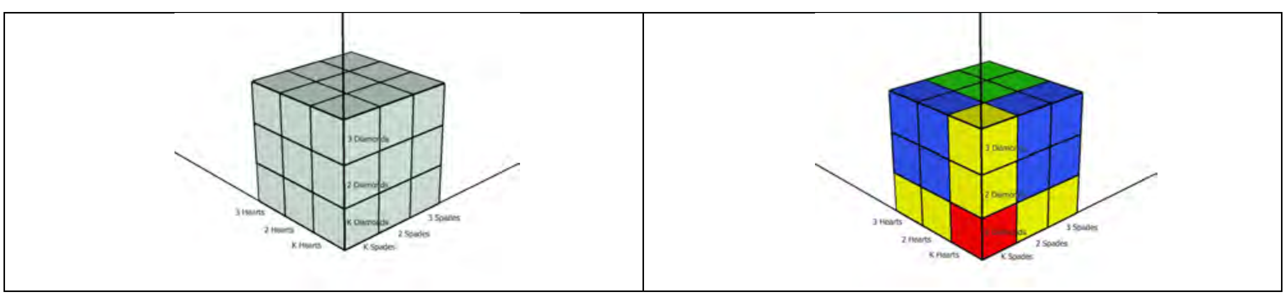

Figure 1a (left), 1b (right). 3-D array highlighting possible types of three-card hands[i]

Empirical Examples: Learning, Generalizing Actions, and Reflection Generalizations

Example one assimilation versus generalizing assimilation. We use Shante's solution of the Subway and Card Problem to illustrate generalizing assimilation. Shante solved the Subway Problem with relative ease she concluded that there should be 24 possible sandwiches.

Subway Problem: Subway has two kinds of bread, three kinds of cheese, and four kinds of meat. A sandwich is one bread, one cheese, and one meat. How many possible sandwiches can subway make?

The interviewer then asked her to determine the total number of possible three card hands in the Card Problem, which included making a number of sample three card hands with actual cards. When she was asked how many possible three card hands she could make, Shante responded quickly that there would be 27 three card hands because the answer would be "three times three times three." However, when asked to explain the multiplication problem and to represent her solution, Shante expressed uncertainty.

I: Why is it that (three times three times three)?
S: Three cards, three hands and three kings? Wait. Okay. Can you say the question again? [The interviewer asks the question again, but Shante is still confused. The interviewer asks Shante to make a tree diagram]
….
S [responding to a question about making a tree diagram for the problem]: Okay. Wait a minute. Okay, now wait. What's going on here? Okay, so this would be a K and a three, and then the rest would branch out from that, right? [writes "K-3-2" on her paper.]
….
S [she has stated that the two in the "K-3-2" is the two of spades, and is responding to the interviewer's question about what the other cards represent]: It will be … a K heart and a three heart … I think that's one hand.

Shante's solution indicated that a critical difference for her in solving the Card Problem was determining what a valid three card hand was—in the Subway Problem it was clear to her that a sandwich could not consist of one bread and two cheeses. However, in the Card Problem it was unclear to her that a three card hand should not consist of one spade and two hearts, despite the fact that she had made three card hands with actual cards and these three card hands contained only one spade, one heart, and one diamond. The primary difference in these two situations seemed to be that each person had the same kind of object, cards, that were differentiated by suit whereas in the Subway Problem the objects were not of the same kind. This led to a perturbation for Shante, "this problem got me all the way messed up", that she eventually resolved by relating the suits of the cards to the Subway Problem, "Could the spades be like the meats?....The diamonds could be like the cheeses?....Oh, the hearts would be like the bread!" Because Shante's abstraction of similarity between the two situations was a primary feature of her solution of the problem, we considered her solution to involve a generalizing assimilation, which was an act of learning that entailed an adjustment to the assimilatory mechanism of her scheme.

Example two recursion of a scheme. Our second example is drawn from a 10th grade student in the interview study and allows us to examine a functional accommodation that involved recursion. Trevon was initially presented with a modification of the Subway Problem that involved four breads and six meats. Trevon said, "In my head I'm attributing one bread to each meat. … and if you do that, it would just be 4 times 6. [He subsequently made the beginning of a tree diagram]." Trevon was able to quickly complete the task and provide a justification for his way of operating, which we take as indication that he assimilated the situation using an extant scheme. The activity of his scheme involved multiple operations, but we highlight that it included a systematic use of his *pairing operation*, an operation that involves a student in creating the sandwich as a unit that contains two units (i.e, a bread and a meat), a pair. This operation can be the basis for establishing the identity that one times one is one, and we focus on it because it is a key feature of students' solutions of combinatorics problems that differentiates them from other multiplicative situations.

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Later in the interviews, Trevon was presented with the Card Problem where each person had three cards. He responded that the number of possible three card hands would be, “eighteen”, explaining, “there’s six combos [indicating hearts and diamonds combined] for each card and there’s three cards total [indicating the three spades]. So, that’s six times three.” Our interpretation of Trevon’s response was that he assimilated the Card Problem using the scheme he used to solve the Subway Problem. There were three spades that could be paired with each of six cards. The interviewer asked him to show the eighteen three card hands with the actual cards. To do so, he initially fixed a spade and simultaneously moved two cards, one heart and one diamond, to be paired with the spade. By doing so he essentially treated the heart and diamond cards as if they were like the meats in the Subway Problem, meaning he did not pair the heart with the diamond first to establish a heart-diamond pair. He experienced a perturbation, however, because he was trying to monitor which three card hands he had created and was not able to do so. His monitoring led him to introduce a novel way of operating: he paired each heart with each diamond, and took the result of this scheme, the nine heart-diamond pairs, as material to operate on with his scheme, pairing each heart-diamond pair with a spade.

We interpret this as a functional accommodation that entailed taking the result of his scheme as input for using his scheme again. For this reason, we consider the functional accommodation to have involved recursion; the recursive process did not involve his use of any novel operations. However, it involved more than just repeated use of the scheme—the result of the first instantiation of the scheme was embedded in the result of the second instantiation of the scheme. We consider this embedding process to be key to recursion. Moreover, we consider this to be a generalizing action because it broadened the class of problems Trevon could solve.

Example three embedding novel operations into the activity of a scheme. As our description of Shante’s activity in Example One suggested she had constructed a scheme for solving problems like the Subway Problem with breads, meats, and cheeses, and vis-à-vis a generalizing assimilation used her scheme to solve the Card Problem. She solved each problem representing the set of outcomes as either a list or as a tree diagram, and the operations she used to solve the problems were comparable to those outlined for Trevon in Example Two. The interviewer had the goal of having students represent the solution of these problems using multiple 2-D arrays, and then a single 3-D array. For the Subway Problem, Shante easily created two 2-D arrays (Figure 2a) to represent the problem. The interviewer asked her what was changing as she “moved horizontally along the x-axis” and she said, “the cheeses”, and responded similarly to a question about, “the meats”.

The interviewer then said that the goal was to think about how Shante could use her two-dimensional arrays to make a three-dimensional array, specifically asking, “what direction could you move for the breads to change?” Shante responded that she thought her listing of the breads in Figure 2a would “just be a title” for each array, and indicated that she was uncertain about what direction she might move in order for the breads to change. To further investigate this issue, the interviewer had Shante make two 2-D arrays using snap cubes (Figure 2b). After significant questioning, Shante figured out that she could stack one array on top of the other, and the interviewer had her make an array for a third bread (Figure 2c). At this point she still had “no idea” in which direction you’d move for the breads to change so the interviewer asked her to imagine where all the sandwiches with bread four, five, six, and seven would be, and for each she responded, “right above” all the sandwiches for the prior bread. The interviewer then asked her “if you had to draw a line to show where the breads would be represented (like the lines she had drawn for the meats and cheese), how would she draw it?” Shante responded, “It would be going up!”, and to show she put a pencil at the vertex of the bread and cheese axis in Figure 2c. From the perspective of mental operations, we consider Shante to have envisioned translating her 2-D array up one unit six times, and from this translation to have

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abstracted an axis on which the breads could be located that were similar to the axes for meats and cheeses.

Figure 2a (left), 2b (center), 2c (right).

To determine what other operations Shante might be using in her production of a 3-D array, the interviewer asked her to locate in Figure 2c where the sandwich with the second bread, second cheese, and third meat would be. Shante initially pointed to the top of the 3-D array where the sandwich with bread three, cheese two, meat three was located, and said, “It’s like in this [pointing downward with her pencil], and then pointed to where the sandwich containing bread two, cheese one, and meat three was located, and said, “It’s like right next to this one [pointing inward with her pencil towards where the cube was located].” She then clarified that by “in this” she meant “underneath” the cube that she had pointed to on top of the array.

Her identification that the cube for bread two, cheese two, meat three was “underneath” the cube that represented bread three, cheese two, and meat three, indicated that as part of translating the layers of her array upward she had also translated the referential system upward (i.e., the meat-cheese axes), and could use that to locate points in her 3-D array. The fact that she also identified the cube for bread two, cheese two, meat three as “next to” the cube for bread two, cheese one, and meat three, indicated that she could envision switching frames of reference; she used the meat-bread axes to locate the correct cube in the “cheese 1” plane, and then envisioned that the correct cube would be translated one unit inward on the 3-D array. We consider her switching frames of reference to be indication that she could mentally rotate the “meat-cheese axes” to become the “bread-meat axes”.

We consider the operations of translation and rotation to have been *embedded* in the activity of her scheme for producing the set of outcomes. Because these operations were not part of her initial solution to the problem, we consider this to have been a functional accommodation to her scheme. We consider this functional accommodation to be a generalizing action because it allowed her to create a spatial structuring for a broader class of objects where her spatial structuring for 2-D arrays was embedded in her spatial structuring for 3-D arrays vis-à-vis the operations of translation and rotation.

Example four operating on the result of a scheme with operations external to the scheme. DeShay entered the interview study with a scheme for solving combinatorics problems like the Outfits and Subway Problem, and with relative ease could represent the set of outcomes using a 3-D array. We took this as indication that her scheme for solving such problems included the operations described in example two and three. Moreover, these operations seemed well established for DeShay, and so we considered that the result of these operations (the set of outcomes represented as a spatially structured 3-D array) was material that she could operate on with operations that were external to her scheme for producing them. To illustrate this issue, we provide data where DeShay was finding the regions of her 3-D array that corresponded to a three card hands that had a certain number of face cards, specifically the three card hands that contained one king where the king was the king of spades.

> D: Okay, so we have…Want to start with the king of spades, so it'd be these-by-these [points to the region in the 3-D array that represents the twenty-five three-card hands that contain the king of spades, Figure 3a], but it wouldn't be any in this row because that's the king of hearts (Figure 3b). So, it's just these four by these four. But it wouldn't be any -- or, these four by these five (Figure 3c). But it wouldn't be any of the very bottom because that's the king of diamonds (Figure 3d). So, it's this four by that four (Figure 3e).

To solve this problem DeShay took the *intersection* of the set of three card hands that contained the king of spades with the set of three card hands (Figure 3a) that contained the king of diamonds (Figure 3b), and *eliminated* this from the original set because those three card hands would contain both the king of spades and king of diamonds (Figure 3c). She then used these operations recursively: she took the *intersection* of this new set ("these four by these five") with the set of three card hands that contained the king of diamonds (Figure 3d), and *eliminated* these three card hands because they contained the king of spades and king of diamonds (Figure 3e).

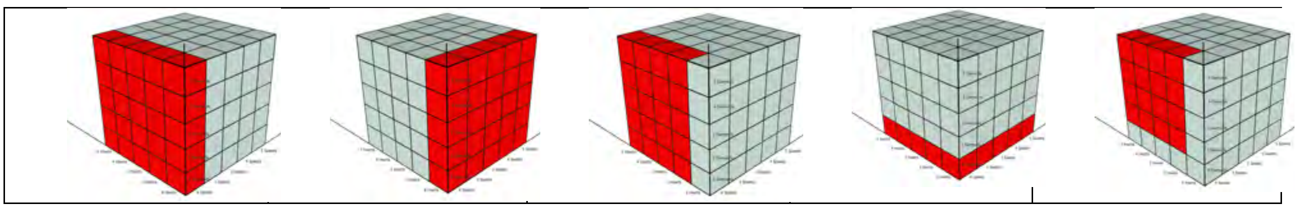

Figure 3a (left), 3b (second from left), 3c (middle), 3d (second from right), 3e (right).

We consider this functional accommodation to entail DeShay using operations external to the ones that produced the set of three card hands represented as a 3-D array. We consider it to be a generalizing action because DeShay's implementation of these operations led to the abstraction of a spatial structuring like the one shown in Figure 1b where she no longer had to implement the operations in order to impose the spatial structuring on future cases of the problem, represented as other cubes (e.g., a seven by seven by seven cube or a one hundred by one hundred by one hundred cube). She could simply assimilate future situations with the result of her operating, a cube that contained eight regions.

Example five reflection generalization involving reflected abstraction. After solving versions of the Card Problem that involved three, four, and five cards, the interviewer asked DeShay to imagine and describe what the cube would look like for the case of six cards, and then asked her if she could write a formula for if there were an unknown number of non-face cards. DeShay first wrote, "$y=1+x$" where y was the total number of cards and x the total number of non-face cards, and then wrote "$(1 + x)^3 = x^3 + (1 \bullet x^2 \bullet 3) + (1^2 \bullet x \bullet 3) + 1^3$." We considered these to be symbols that DeShay could use to call forth the operations outlined in examples two through four without actually having to implement these operations in full. We make this inference based on explanations like the following of how she got "$(1 \cdot x^2 \cdot 3)$":

> D: There would be one option times …Oh. Can't exactly put the multiplication sign. Times x options, also times it by x options again, because with one suit you'd only have one king [points to where the king of spades is represented in the 3-D array] and the other ones you would have

….

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> x different cards to pick from, so x times x. And then you'd also multiply it by three for the three suits. Then you get one times x squared times three.

Her explanation clearly referred back to the Card Problem. Throughout these explanations, she gestured to the five by five by five cube, showing imagined locations of where particular regions would be in the general case. We took this as indication that the mathematical symbols she could use to stand in for operations that could be implemented, but no longer needed to be, and which were closely tied to re-presentations of her activity in prior cases of the problem.

We considered this to be a reflection generalization based on reflected abstraction because it was a final formal statement of generalization that grew from a retroactive thematization of her reasoning. DeShay stated as much when talking about Figure 5b, saying "So, this will be the formula to solve because this is pretty much similar to all these other ones [points to the cases of three, four, five, and six cards in her chart], except we had numbers to plug into them."

Discussion

One central contribution of this work is that our study of generalizing is situated within a framework for learning, which allows for the identification of different processes involved in generalizing. We do not consider our examples to be comprehensive, but rather illustrative. The first example illustrated how a student broadened the assimilatory mechanism of her scheme; the second example illustrated a student that recursively used his scheme, taking the result of a scheme and operating on it with operations that were internal to the scheme; the third example illustrated a student who embedded novel operations into an already extant scheme; and the fourth example illustrated a student who took the result of a prior scheme (a 3-D array) and operated on it with operations that were external to the scheme. Beyond just characterizing these different processes, situating the study in a framework for learning allows for a clear distinction between statements and actions that a student makes that are general, but involve no novel ways of operating (i.e., are based on extant schemes), from those that do involve novel ways of operating (i.e., that involve learning). The final example then shows the connection between the operational structures that students developed during the interview study and their expression in symbolic form. As such the paper provides an exemplar of how symbols function in a way that is compatible with Von Glasersfeld's (1995) model of this process.

Endnotes

[i] Students created an array with snap cubes that were all the *same* color like Figure 1a. We used snap cubes because we could find no good way to physically or virtually represent a 3-D discrete array.

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